Localization of Eigenstates & Mean Wehrl Entropy

Karol Życzkowski[1,2]
[1] *Instytut Fizyki im. Mariana Smoluchowskiego, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 Kraków, Poland*
[2] *Centrum Fizyki Teoretycznej PAN Al. Lotników 32/44, 02-668 Warszawa, Poland*
(October 22, 1999)

Dynamics of a periodically time dependent quantum system is reflected in the features of the eigenstates of the Floquet operator. Of the special importance are their localization properties quantitatively characterized by the eigenvector entropy, the inverse participation ratio or the eigenvector statistics. Since these quantities depend on the choice of the eigenbasis, we suggest to use the overcomplete basis of coherent states, uniquely determined by the classical phase space. In this way we define the mean Wehrl entropy of eigenvectors of the Floquet operator and demonstrate that this quantity is useful to describe quantum chaotic systems.

e-mail: [1]karol@tatry.if.uj.edu.pl

I. INTRODUCTION

Analysis of quantum chaotic systems is often based on the statistical properties of the spectrum of the Hamiltonian H (in the case of autonomous systems) or the Floquet operator F (in the case of periodically perturbed systems). In general, quantized analogous of classically chaotic systems display spectral fluctuations conforming to the predictions of random matrices. Depending on the geometrical properties of the system one uses orthogonal, unitary or symplectic ensemble [1,2].

Another line of research deals with eigenstates of the analyzed quantum system. One is interested in their localization properties, which can be characterized by the eigenvector distribution [3–6] the entropic localization length [7] or the inverse participation ratio [8]. All this quantities, however, are based on the expansion of an eigenstate in a given basis $\{\vec{b}_i\}$, which may be chosen arbitrarily. If one chooses (with a bad will), $\{\vec{b}_i\}$ as the eigenbasis of F, all these quantities carry no information whatsoever. One may ask, therefore, to what extend the quantitative analysis based on the eigenvector statistics is reliable.

Let G denote a unitary operator, such that $\{\vec{b}_i\}$ is its eigenbasis. We showed [9–11] that the eigenvector statistics of a quantum map F conforms to the prediction of random matrices, if operators F and G are *relatively random*, i.e., their commutators are sufficiently large.

In this paper we advocate an alternative method of solving the problems with arbitrariness of the choice of the expansion basis. Instead of working in a finite discrete basis, we shall use the coherent states expansion of the eigenstates of F. For several examples of compact classical phase spaces one may construct a canonical family of the generalized coherent states [12]. Localization properties of any pure quantum state may be characterized by the Wehrl entropy, equal to the average log of its overlap with a coherent state [13,14]. We propose to describe the structure of a given Floquet operator F by the mean Wehrl entropy of its eigenstates. This quantity, explicitly defined without any arbitrariness, is shown to be a useful indicator of quantum chaos.

This paper is organized as follows. In section II we review the definition of the Husimi distribution, stellar representation, and the Wehrl entropy. For concreteness we work with the $SU(2)$ vector coherent states, linked to the algebra of the angular momentum operator and corresponding to the classical phase space isomorphic with the sphere. In section III we define the mean Wehrl entropy of eigenstates and present analytical results obtained for low dimensional Hilbert spaces. Exemplary application of this quantity to the analysis of the quantum map describing the model of the periodically kicked top is provided in section IV.

II. HUSIMI DISTRIBUTION AND STELLAR REPRESENTATION

Consider a compact classical phase space Ω, a classical area preserving map $\Theta : \Omega \to \Omega$ and a corresponding quantum map F acting in an N-dimensional Hilbert space $\mathcal{H}_N$. A link between classical and quantum mechanics can be established via a family of generalized coherent states $|\alpha\rangle$. For several examples of the classical phase spaces there exist a canonical family of coherent states. It forms an overcomplete basis and allows for an identity resolution $\int_\Omega |\alpha\rangle\langle\alpha| d\alpha = \mathbf{1}$. Any mixed quantum state, described by a density matrix ρ can be represented by the generalized Husimi distribution [15], (Q-function)

$$H_\rho(\alpha) := \langle\alpha|\rho|\alpha\rangle. \tag{2.1}$$

The standard normalization of the coherent states, $\langle\alpha|\alpha\rangle = 1$, assures that $\int_\Omega H_\rho(\alpha) d\alpha = 1$. For a pure quantum state $|\psi\rangle$ the Husimi distribution is equal to the overlap with a coherent state $H_\psi(\alpha) := |\langle\psi|\alpha\rangle|^2$. Let us note that the Husimi distribution was successfully applied to study dynamical properties of quantized chaotic systems [16,17].

Consider a discrete partition of the unity into n terms; $\sum_{i=1}^n p_i = 1$. The Shannon entropy $S_d = -\sum_{i=1}^n p_i \ln p_i$ characterizes uniformity of this partition. In an analogous way one defines the Werhl entropy of a quantum state ρ [13]

$$S_\rho = -\int_\Omega H_\rho(\alpha) \ln[H_\rho(\alpha)] d\alpha, \tag{2.2}$$

in which the summation is replaced by the integration over the classical space Ω. This quantity characterizes the localization properties of a quantum state in the classical phase space. It is small for coherent states localized in the classical phase space Ω and large for the delocalized states. The maximal Wehrl entropy corresponds to the maximally mixed state ρ_*, proportional to the identity matrix, for which the Husimi distribution is uniform.

Although the notions of the generalized coherent states, the Husimi distribution, and the Wehrl entropy are well defined for several classical compact phase spaces, in this work we analyze in detail only the most important case $\Omega = S^2$. This phase space is typical to physical problems involving spins, due to the algebraic properties of the angular momentum operator J. In this case one uses the family of spin coherent states $|\vartheta, \varphi\rangle$ localized at points (ϑ, φ) of the sphere S^2. These states, also called $SU(2)$ vector coherent states, were introduced by Radcliffe [18] and Arecchi *et al.* [19] and are an example of the general group theoretic construction of Perelomov [12].

Consider an $N = 2j + 1$ dimensional representation of the angular momentum operator J. For a reference state one usually takes the maximal eigenstate $|j, j\rangle$ of the component J_z. This state, pointing toward the "north pole" of the sphere, enjoys the minimal uncertainty. The vector coherent state represents the reference state rotated by the angles ϑ and φ. Its expansion in the basis $|j, m\rangle$, $m = -j, \dots, +j$ reads [20]

$$|\vartheta, \varphi\rangle = \sum_{m=-j}^{m=j} \sin^{j-m}(\frac{\vartheta}{2}) \cos^{j+m}(\frac{\vartheta}{2}) \times \exp\Big(i(j-m)\varphi\Big) \Big[\binom{2j}{j-m}\Big]^{1/2} |j, m\rangle, \tag{2.3}$$

where $\int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta d\vartheta |\vartheta, \varphi\rangle\langle\vartheta, \varphi|(2j+1)/4\pi = \mathbf{1}$.

For the $SU(2)$ coherent state the distribution (2.1) equals in this case $H_\psi(\vartheta, \varphi) := |\langle\psi|\vartheta, \varphi\rangle|^2$. Two different spin coherent states overlap unless they are directed into two antipodal points on the sphere. The Husimi representation of a spin coherent state has thus one zero (degenerated $N-1$ times) localized at the antipodal point. In general, any pure quantum state can be uniquely described by the set of $N-1$ points distributed over the sphere. Some of these zeros may be degenerated, just as in the case of a coherent state. This method of characterizing a pure quantum state is called the stellar representation [21,22].

In the analyzed case of the classical phase space isomorphic with the sphere S^2 the Wehrl entropy (2.2) of a state ρ equals

$$S_\rho = -\frac{2j+1}{4\pi} \int_0^\pi \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi H_\rho(\vartheta, \varphi) \ln\Big[H_\rho(\vartheta, \varphi)\Big], \tag{2.4}$$

since the measure element $d\alpha$ is equal to $(2j + 1) \sin\vartheta d\vartheta d\varphi / 4\pi$. Under this normalization the entropy of the maximally mixed state ρ_* equals to $\ln N$.

The Husimi distribution of an eigenstate $|j, m\rangle$ may be computed directly from the definition (2.3). Due to the rotational symmetry it does not depend on the azimuthal angle φ

$$H_{|j,m\rangle}(\vartheta) = \sin^{2(j-m)}(\vartheta/2) \cos^{2(j+m)}(\vartheta/2) \binom{2j}{j-m}, \tag{2.5}$$

which simplifies the computation of the Wehrl entropy. Simple integration gives for the reference state $|j, j\rangle$

$$S_{\rm coh} = \frac{N-1}{N} = \frac{2j}{2j+1}. \tag{2.6}$$

Due to the rotational invariance the Wehrl entropy is the same for any other coherent state.

N	j	m	$S_{\|j,m\rangle}$	$\bar{S}_{J_z}$
2	1/2	1/2	1/2	$1/2 = 0.5$
3	1	1	2/3	$1 - \frac{\ln 2}{3} \approx 0.769$
		0	$5/3 - \ln 2$	
4	3/2	3/2	3/4	$\frac{3}{2} - \frac{\ln 3}{2} \approx 0.951$
		1/2	$9/4 - \ln 3$	
5	2	2	4/5	$2 - \frac{\ln 96}{5} \approx 1.087$
		1	$79/30 - \ln 4$	
		0	$47/15 - \ln 6$	
6	5/2	5/2	5/6	$\frac{5}{2} - \frac{1}{3}\ln 50 \approx 1.196$
		3/2	$35/12 - \ln 5$	
		1/2	$15/4 - \ln 10$	

Table 1. Wehrl entropy $S_{|j,m\rangle}$ for the eigenstates of J_z and its mean $\bar{S}_{J_z}$ for $N = 2, 3, 4, 5, 6$. Due to the geometrical symmetry $S_{|j,m\rangle} = S_{|j,-m\rangle}$.

The Wehrl entropies for other eigenstates of J_z are collected in Tab. 1 for some lower values of N. These results may be also obtained from the general formulae provided by Lee [23] for the Wehrl entropy of the pure states in the stellar representation. Eigenstate $|j, m\rangle$ is represented by $j+m$ zeros at the south pole and $j-m$ zeros at the north poles. For $j = 1/2$ ($N = 2$) all the states are $SU(2)$ coherent, so their entropies are equal. For $j = 1$ ($N = 3$)

the coherent state $|1,1\rangle$ is characterized by the smallest entropy, while the state $|1,0\rangle$ by the largest (among the pure states). The larger N, the more place for a various behaviour of pure states, measured by the values of S. The axis of the Wehrl entropy is drawn schematically in Fig.1.

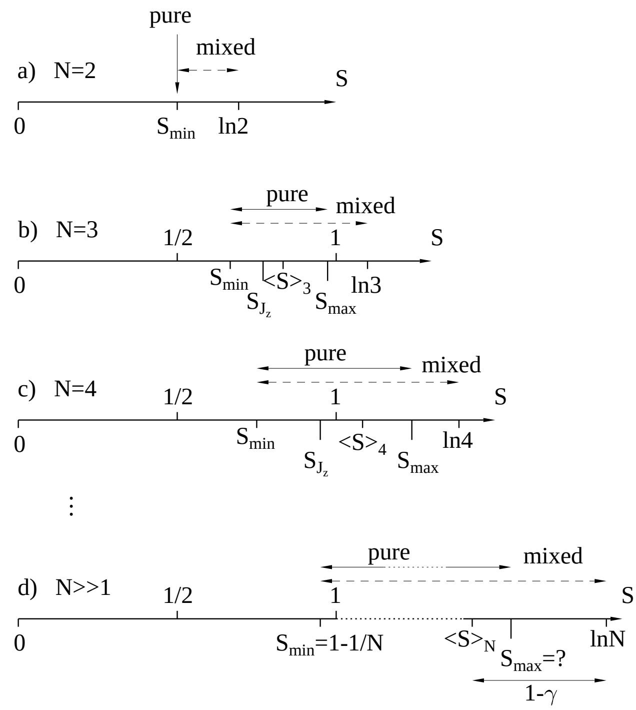

FIG. 1. Axis of Wehrl entropy for pure states of N dimensional Hilbert space; a) $N=2$ for which $\bar{S}_{\min}=\bar{S}_{\max}=1/2$; b) $N=3$ for which $\bar{S}_{\min}=2/3$, $\bar{S}_{J_z}\approx 0.77$, $\langle S\rangle_3\approx 0.83$, $S_{\max}\approx 0.973$; c) $N=4$ for which $\bar{S}_{\min}=3/4$, $\bar{S}_{J_z}\approx 0.95$, $\langle S\rangle_4\approx 1.08$, $S_{\max}\approx 1.24$; and d) $N>>1$, where $\bar{S}_{\min}=1-1/N$ while $\langle S\rangle_N\approx \ln N-0.423$.

It has been conjectured by Lieb [24] that vector coherent states are characterized by the minimal value of the Wehrl entropy $S_{\min}=S_{\rm coh}$, the minimum taken over all mixed states. For partial results in the direction to prove this conjecture see [25,23,26]. It was also conjectured [23] that the states with possibly regular distribution of all $N-1$ zeros of the Husimi function on the sphere are characterized by the largest possible Wehrl entropy among all pure states $S_{\max}$. Such a distribution of zeros is easy to specify for $(N-1)=4,6,8,12,20$, which correspond to the Platonian polyhedra. For $N=2$ all pure states are coherent, so $S_{\min}=S_{\max}=1/2$. For $N=3$ the maximal Wehrl entropy $S_{\max}=5/3-\ln 2\approx 0.97$ is achieved for the state $|1,0\rangle$, for which the two zeros of Husimi function are localized at the opposite poles of the sphere. For $N=4$ the state with three zeros located at the equilateral triangle inscribed in a great circle is characterized by $S_{\max}=21/8-2\ln 2\approx 1.24$. It will be interesting to find such maximally delocalized pure states for larger values of N, and to study the dependence $S_{\max}$ of N.

Let us emphasize that for $N>>1$ the pure states exhibiting small Wehrl entropy, (of the order of $S_{\min}$), are not typical. In the stellar representation coherent states correspond to coalescence of all $N-1$ zeros of Husimi distribution in one point. In a typical situation the density of the zeros is close to uniform on the sphere, and the Wehrl entropy of such delocalized pure states is large. A random state can be generated according to the natural uniform masure on the space of pure states by taking any vector of a $N\times N$ random matrix distributed according to the Haar measure on $U(N)$. Averaging over this measure one may compute the mean Wehrl entropy $\langle S\rangle_N$ of the pure states belonging to the N dimensional Hilbert space. Such integration was performed in [3,27,28] in a slightly different context leading to

$$\langle S\rangle_N=\Psi\left(N+1\right)-\Psi\left(2\right)=\sum_{n=2}^{N}\frac{1}{n}, \qquad (2.7)$$

where Ψ denotes the digamma function. Note that another normalization of the coherent states used in Ref. [28], leads to results shifted by a constant $-\ln N$. Such a normalization allows one for a direct comparison between the entropies describing the states of various N. In the asymptotic limit $N\to\infty$ the mean entropy $\langle S\rangle_N$ behaves as $\ln N+\gamma-1\sim\ln N-0.42278$, which is close to the maximal possible Wehrl entropy for mixed states $S_{\rho_*}=\ln N$. This result is schematically marked in Fig 1d.

III. MEAN WEHRL ENTROPY OF EIGENSTATES OF QUANTUM MAP

Consider a quantum pure state in the N-dimensional Hilbert space. Its Wehrl entropy computed in the vector coherent states representation may vary from $1-1/N$, for a coherent state, to the number of order of $\ln N$, for the typical delocalized state. This difference suggests a simple measure of localization of eigenstates of a quantum map F. Denoting its eigenstates by $|\psi_i\rangle$; $i=1,\dots,N$ we define the mean Wehrl entropy of eigenstates

$$\bar{S}_F=\frac{1}{N}\sum_{i=1}^{N}S_{|\psi_i\rangle}. \qquad (3.1)$$

This quantity may be straightforwardly computed numerically for an arbitrary quantum map F. For quantum analogues of classically chaotic systems exhibiting no time reversal symmetry all eigenstates are delocalized. In this case the mean Wehrl entropy of eigenvectors $\bar{S}_F$ fluctuates around $\langle S\rangle_N\sim\ln N$.

In the opposite case of an integrable dynamics the eigenstates are, at least partially, localized. A simple example is provided by any Hamiltonian diagonal in the

J_z basis (or the basis of any other component of J). The mean Werhl entropy $\bar{S}_{J_z}$ is given in table 1 for some values of N. Further analysis shows [31] that for larger N the mean entropy behaves as $\frac{1}{2}\ln N$. This result has a simple interpretation. Let us divide the surface of the sphere into $N \sim \hbar^{-1}$ cells. A typical eigenstate of J_z is localized in a longitudinal strip of a constant polar angle θ and covers $\sqrt{N}$ of the cells, so its entropy is of the order of $\ln\sqrt{N}$.

The quantity $\bar{S}$ is well–defined in a generic case of operators F with a nondegenerate spectrum. In the case of the degeneracy, there exists a freedom of choosing the eigenvectors; to cure this lack of uniqueness we define $\bar{S}_F$ as the minimum over all possible sets of eigenvectors of F. Having a general definition of the mean Wehrl entropy of eigenvectors of an arbitrary unitary operator, one may pose a question, for which operators $F_{\rm min}$ ($F_{\rm max}$) of a fixed N with a nondegenerate spectrum this quantity is the smallest (the largest). It is clear that $\bar{S}_{F_{\rm min}}$ is larger than $S_{\rm coh}$ (for $N > 2$), since the set of any N coherent states does not form an orthogonal basis. On the other hand, the minimum is smaller than $\bar{S}_{J_z}$, as explicitly demonstrated in Appendix for $N = 3$. The value $\bar{S}_{F_{\rm max}}$ is larger than the average over the random unitary matrices $\langle S\rangle_{U(N)} = \langle S\rangle_N$ and smaller than $S_{\rho_*} = \ln N$.

The mean Werhl entropy of the eigenstates $\bar{S}_F$ may be related with the eigenvector statistics of the operator F. Let us expand a given coherent state in the eigenbasis of the Floquet operator, $|\alpha\rangle = \sum_{i=1}^{N} c_i(\alpha)|\psi_i\rangle$. The dynamical properties of a quantum system are characterized locally [29] by the Shannon entropy $S_s(\alpha) := -\sum_{i=1}^{N} |c_i(\alpha)|^2 \ln|c_i(\alpha)|^2$. The mean Wehrl entropy may be thus written as an average over the phase space

$$\bar{S}_F = \frac{1}{N}\int_\Omega S_s(\alpha)d\alpha. \tag{3.2}$$

This link is particularly useful to analyze the influence of the time reversal symmetry. In presence of any generalized antiunitary symmetry the operator F may be described by the circular orthogonal ensemble (COE). There exists a symmetry line in the phase space and the coherent states located along this line display eigenvector statistics typical of COE [32]. This symmetry is also visible in the stellar representation of the eigenstates and manifests itself by a clustering of zeros of Husimi functions [33,34]. However, a typical coherent state does not exhibit such a symmetry and its eigenvector statistics is typical to the circular unitary ensemble (CUE). Thus for a system with the time-reversal symmetry the mean Wehrl entropy will be slightly smaller than for the analogous system with the time reversal symmetry broken, but much larger than the Shannon entropy of real eigenvectors of matrices pertaining to the orthogonal ensemble.

IV. MEAN WEHRL ENTROPY FOR THE KICKED TOP

In order to demonstrate usefulness of the mean Wehrl entropy in the analysis of quantum chaotic systems we present numerical results obtained for the periodically kicked top. This model is very suitable for investigation of quantum chaos [35,1]. Classical dynamics takes place on the sphere, while the quantum map is defined in terms of the components of the angular momentum operator J. The size of the Hilbert space is determined by the quantum number j and equals $N = 2j + 1$. One step evolution operator reads $F_o = \exp(-ipJ_z)\exp(-ikJ_x^2/2j)$. For $p = 1.7$ the classical system becomes fully chaotic for the kicking strength $k \approx 6$ [35]. This system possesses a generalized antiunitary symmetry and can be described by the orthogonal ensemble. The time reversal symmetry may be broken by an additional kick [1]. The system $F_u = F_o\exp(-ik'J_y^2/2j)$ pertains to CUE and will be called the unitary top.

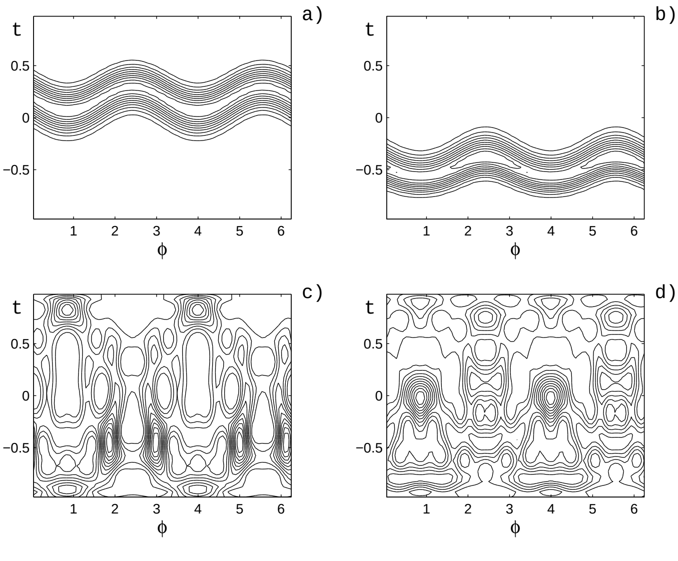

FIG. 2. Husimi distribution of exemplary eigenstates of the Floquet operator of the orthogonal kicked top for $N = 62$ in the dominantly regular regime ($k = 0.5$), a) and b), and chaotic regime ($k = 8.0$), c) and d). The sphere is represented in a rectangular projection with $t = \cos\vartheta$.

Fig. 2 presents the Husimi distributions of two eigenstates of F_o for $p = 1.7$ in the regime of regular motion ($k = 0.5$) and two, for which the classical dynamics is chaotic ($k = 8.0$). In the quasiregular case the eigenstates are localized close to parallel strips, covered uniformly by the eigenstates of J_z. On the other hand, the eigenstates of the chaotic map are delocalized at the entire sphere. These differences are well characterized by the values of the Wehrl entropies, equal correspondingly: a) 2.77, b) 2.66; and c) 3.72, d) 3.80. The data, obtained for $N = 62$, may be compared with the mean entropy of the unperturbed system, $\bar{S}_{J_z} \approx 2.465$, the mean Wehrl entropy of chaotic system without time reversal symetry, $\langle S \rangle_{62} \approx 3.712$, and the maximal entropy of the mixed state, $S_{\rho_*} = \ln 62 \approx 4.1271$.

The above eigenstates are typical for both systems, and the other 60 states display a similar character. The properties of all eigenstates are thus described by the mean Wehrl entropy of eigenstates $\bar{S}_F$. The dependence of this quantity on the kicking strength k is presented in Fig. 3. To show a relative difference between the entropies typical to the regular dynamics we use the scaled coefficient

$$\mu(F) := \frac{\bar{S}_F - \bar{S}_{J_z}}{\langle S \rangle_N - \bar{S}_{J_z}}. \tag{4.1}$$

Per definition μ is equal to zero, if F is diagonal in the J_z basis, which corresponds to the integrability. In the chaotic regime F is well described by CUE and μ is close to unity. This is indeed the case for the unitary top with $k' = k/2$ and $k > 6$. The growth of μ is bounded and therefore it cannot, in general, follow the increase of the classical Kolmogorov–Sinai entropy Λ (the Lapunov exponent averaged over the phase space), which for the classical system grows with the kicking strength k [11]. The data for the orthogonal top fluctuate below unity, due to existence of the symmetry line. The difference between the coefficients μ obtained for both models does not depend on the kicking strength, but decreases with N and vanish in the classical limit $N \to \infty$.

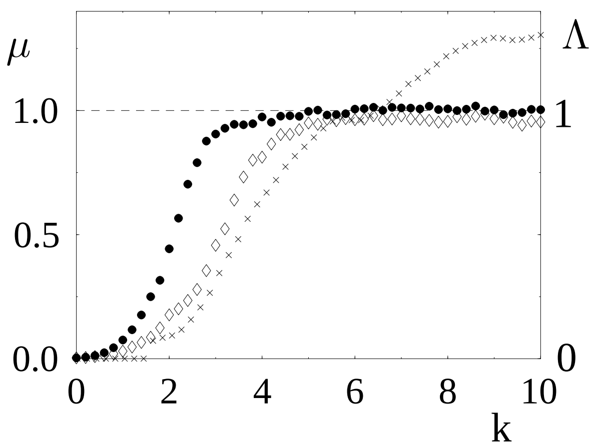

FIG. 3. Scaled mean Wehrl entropy μ of the eigenvectors of the Floquet operator for the kicked top as a function of the kicking strength k for $N = 62$. The data are obtained for two models: unitary top (•) and orthogonal top (⋄). The crosses denote values of the classical Kolmogorov-Sinai entropy Λ, which characterize the transition to chaos in the classical analogue of the orthogonal top.

V. CONCLUDING REMARKS

The Wehrl entropy of a given state characterizes its localization in the classical phase space. We have shown that the mean Wehrl entropy $\bar{S}_F$ of eigenstates of a given evolution operator F may serve as a useful indicator of quantum chaos. Let us emphasize that this quantity, linked to the classical phase space by a family of coherent states, does not depend on the choice of basis. This contrasts the others quantities, like eigenvector statistics, localization entropy, inverse participation ratio, often used to study the properties of eigenvectors. It will be interesting to find the unitary operators (or rather the repers) for which $\bar{S}_F$ is the smallest or the largest.

The mean Wehrl entropy of eigenstates enables one to detect the transition from regular motion to chaotic dynamics. On the other hand, it is not related to the classical Kolmogorov–Sinai entropy (or to the Lapunov exponent), so it cannot be used to measure the degree of chaos in quantum systems. Such a link with the classical dynamics is established for the *coherent states dynamical entropy* of a given quantum map [36,28], but this quantity is much more difficult to calculate. Both these quantities characterize the *global* dynamical properties of a quantum system, in contrast to the entropy of Mirbach and Korsch [37,38], which describes the *local* features.

Mean Wehrl entropy characterizes the structure of eigenvectors of F, and is not related at all to the spectrum of this operator. Thus it is possible to construct a unitary operator with a Poissonian spectrum and the delocalized eigenvectors. Or conversely, one may find an operator with spectrum typical to CUE and all eigenstates localized. This shows that the relevant information concerning the dynamical properties of a quantum system described by an unitary evolution operator F is contained as well in its spectrum and in its eigenstates.

I am indebted to W. Słomczyński for fruitful discussions and a constant interest in the progress of this research. I am also thankful to M. Kuś and P. Pakoński for helpful remarks. It is a pleasure to thank Bernhard Mehlig for the invitation to Dresden and the Center for Complex Systems for a support during the workshop. Financial support from Polski Komitet Badań Naukowych in Warsaw under the grant no 2P-03B/00915 is gratefully acknowledged.

APPENDIX A: MINIMAL MEAN WERHL ENTROPY FOR $N = 3$

In the case $j = 1$ ($N = 3$) any pure state may be described by the position of two points on the sphere - zeros of the corresponding Husimi distribution. In the stellar representation the eigenstates $|m\rangle$ of J_z are described by $1 + m$ zeros at the south pole of the sphere, and $1 - m$ zeros at the north pole, with $m = -1, 0, 1$. The mean Werhl entropy of eigenstates of J_z is equal to

$1-(\ln 2)/3 \approx 0.769$. We find another set of three orthogonal pure states, characterized by a smaller value of $\bar{S}$, by allowing to move one of the zeros of the Husimi representation along the meridian $\varphi = 0$ (and $\varphi = \pi$). In other words, let us define two orthogonal states

$$\begin{aligned} |\chi_+\rangle &:= \cos\chi|1\rangle + \sin\chi|0\rangle, \\ |\chi_-\rangle &:= -\sin\chi|1\rangle + \cos\chi|0\rangle. \end{aligned} \tag{A1}$$

Keeping the state $|-1\rangle$ fixed we define thus a one parameter family of orthogonal basis $O_\chi = \{|-1\rangle, |\chi_-\rangle, |\chi_+\rangle\}$. The Husimi distributions of the state $|\chi_-\rangle$ has two zeros at the polar angles $\{\theta_-, \pi\}$, while the zeros of $|\chi_+\rangle$ are located at $\{\pi, 2\pi - \theta_+\}$. Here $\theta_- = (2-c^2)/(2+c^2)$ and $\theta_+ = (2c^2-1)/(2c^2+1)$ with $c = \tan\chi$. The angle θ belongs to $[0, 2\pi)$ and the $\theta = \pi$ represents the south pole, while $\theta = 0$ (or $\theta = 2\pi$) denotes the north pole. The location of zeros of Husimi representation of eigenstates of J_z and $O_{\pi/4}$ is presented in Fig.4.

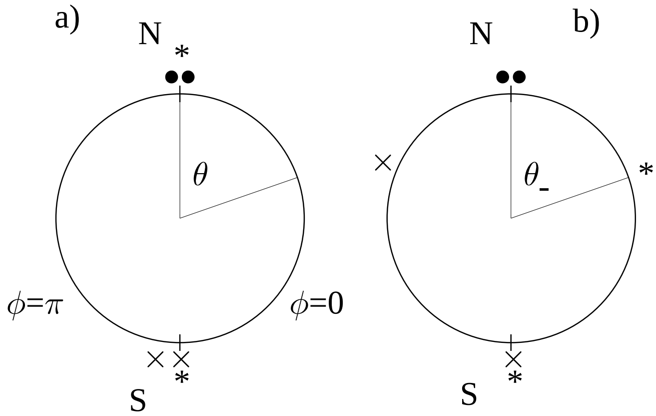

FIG. 4. Cross-section of the Bloch sphere along the meridian $\phi = 0$ ($\phi = \pi$). a) zeros of the eigenstates of the operator J_z: $|-1\rangle$ ($\bullet$), $|0\rangle$ ($*$), and $|1\rangle$ ($\times$); b) zeros of the eigenstates of $O_{\pi/4}$: $|-1\rangle$ ($\bullet$), $|\chi_-\rangle$ ($*$), and $|\chi_+\rangle$ ($\times$).

In the paper of Lee [23] one finds the Wehrl entropy of a $N = 3$ state expressed as a function of the angle ω between both zeros of the Husimi distribution:

$$S = \frac{2/3 + \sigma/6}{1-\sigma/2} + \ln\Big(1 - \frac{\sigma}{2}\Big), \tag{A2}$$

where $\sigma = \sin^2(\omega/2)$. The equivalent result was earlier established by Scutaru [25], $S = 2/3 + [a - \ln(1+a)]$, where $a = \sigma/(2-\sigma)$. Substituting for ω the angles $\pi - \theta_\pm$ we get explicit formulae for the Wehrl entropy of each state $|\chi_\pm\rangle$, which allow us to write the mean entropy $\bar{S}_{O_\chi}$ as a function of the parameter χ. The minimum is achieved for $\chi = \pi/4$ (so $c = 1$ and $\cos\theta_- \cos\theta_+ = 1/3$) and reads $\bar{S}_{O_{\pi/4}} = 1 - [\ln(9/4)]/3 \approx 0.730$. This results shows that the eigenbasis of J_z does not provide the reper characterized by the minimal mean Wehrl entropy, but it does not solve the problem of finding the global minimum. One can expect, that this minimum will occur for a set of N mutually orthogonal states, each of them as close to the coherent state, as possible.

[1] F. Haake, Quantum Signatures of Chaos, Springer, Berlin, 1991.
[2] G. Casati and B. Chirikov (eds), Quantum Chaos. Between Order and Disorder, Cambridge University Press, Cambridge, 1995.
[3] M. Kuś, J. Mostowski, and F. Haake, J. Phys. A 21 (1998) L1073.
[4] O. Bohigas, in *Chaos and Quantum Physics*, eds. M.-J. Giannoni, A. Voros, and J. Zinn-Justin, Les Houches Summer School, Session LII, North-Holland, Amsterdam, 1991.
[5] F. Izrailev, Phys. Rep. 129 (1990) 299.
[6] F. Haake and K. Życzkowski, Phys. Rev. A 42 (1990) 1013.
[7] G. Casati, I. Guarneri, F. Izrailev, and R. Scharf, Phys. Rev. Lett. 64 (1990) 5.
[8] E. Heller, Phys. Rev. A 35 (1987) 1360.
[9] M. Kuś and K. Życzkowski, Phys. Rev. A 44 (1991) 956.
[10] K. Życzkowski, Operator Theory: Adv. & Appl. 70 (1994) 277.
[11] K Życzkowski, Acta Phys. Pol. B 24 (1993) 967.
[12] A. Perelomov, Generalized Coherent States and their Applications, Springer, Berlin, 1986.
[13] A. Wehrl, Rev. Mod. Phys. 50 (1978) 221.
[14] A. Wehrl, Rep. Math. Phys. 30 (1991) 119.
[15] K. Husimi, Proc. Phys. Math. Soc. Jpn. 22 (1940) 264.
[16] K. Takahashi, J. Phys. Soc. Jap. 55 (1986) 762.
[17] K. Życzkowski, Phys. Rev. A 35 (1987) 3546.
[18] J. M. Radcliffe, J. Phys. A 4 (1971) 313.
[19] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6 (1972) 2211.
[20] V. R. Vieira and P. D. Sacramento, Ann. Phys. (N.Y.) 242 (1995) 188.
[21] R. Penrose, Emperor's New Mind, Oxford University Press, Oxford, 1989.
[22] P. Lebœuf and A. Voros, J. Phys. A 23 (1990) 1765.
[23] C-T. Lee, J. Phys. A 21 (1988) 3749.
[24] E. H. Lieb, Comm. Math. Phys. 62 (1978) 35.
[25] H. Scutaru, preprint FT-180 (1979) and Lanl e-print math-ph/9909024.
[26] P. Schupp, Lanl e-print math-ph/9902017.
[27] K. R. W. Jones, J. Phys. A 24 (1991) 121.
[28] W. Słomczyński and K. Życzkowski, Phys. Rev. Lett. 80 (1998) 1880.
[29] K. Życzkowski, J. Phys. A 23 (1990) 4427.
[30] W. K. Wootters, Found. Phys. 20 (1990) 1365.
[31] W. Słomczyński, J. Kwapień, and K. Życzkowski, unpublished.
[32] K. Życzkowski, in Quantum Chaos, ed. H. Cerdeira, World Scientific, Singapore, 1991.

[33] E. Bogomolny, O. Bohigas, and P. Lebœuf, Phys. Rev. Lett. 68 (1992) 2726.
[34] D. Braun, M. Kuś, and K. Życzkowski, J. Phys. A 30 (1997) L117.
[35] F. Haake, M. Kuś, and R. Scharf, Z. Phys. B 65 (1987) 381.
[36] W. Słomczyński and K. Życzkowski, J. Math. Phys. 35 (1994) 5674; 36 (1995) 5201(E).
[37] B. Mirbach and H. J. Korsch, Phys. Rev. Lett. 75 (1995) 362.
[38] B. Mirbach and H. J. Korsch, Ann. Phys. (N.Y.) 265 (1998) 80.